Two Simple Examples in Non-Euclidean Geometry

by

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Introduction. The idea of taxicab geometry comes from the every day experience of driving a car from one location to another location in the city. Mathematically, we may consider the two locations as points $A(a_1, a_2)$ and $B(b_1, b_2)$. Certainly, when we go to location $B$ from $A$, we must stay on the streets and cannot cut diagonally across any block. Thus Euclidean distance,

$$d_E(A, B) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2},$$

cannot be used in solving this problem. On the contrary, the taxicab distance,

$$d_T = |b_1 - a_1| + |b_2 - a_2|,$$

can realistically reflect the actual distance travelled.

As an example, let $A(1, 1)$ and $B(6, 8)$ be two points. Then $d_E(A, B) = \sqrt{74}$ and $d_T(A, B) = |6 - 1| + |8 - 1| = 12$. Referring to Figures 1 and 2, one can see that Figure 2 represents the actual distance travelled.

Although taxicab geometry is almost the same as Euclidean coordinate geometry, it is shown in this paper by two simple examples that the former is in fact classified as non-Euclidean geometry, which is a geometry whose marked characteristic is its rejection of Euclid's parallel postulate as stated in [1].

Although Euclid's postulate is well known to geometry teachers, for completeness sake we would still include it as statement (1) in the following equivalent statements found in Euclidean geometry. Interested readers may refer to [2] for additional information.

(1) If a transversal cuts two lines so that the angles formed on one side of the transversal is less than two right angles, then the two lines will meet on that same side.

(2) Through a point not on a given line
there passes not more than one parallel to the line.

3. Two lines that are parallel to the same line are parallel to each other.

4. A line that meets one of two parallels also meets the other.

5. Parallel lines are equidistant from one another.

**Main Results.** First, we are going to examine the definition of a line in Euclidean geometry and taxicab geometry.

In [3], Pınar pointed out that the most natural way of defining a line in geometry is to consider it as the locus (set of points) which is equidistant from two given points, say \( P_1 \) and \( P_2 \). For instance, a line in Euclidean geometry can be viewed as the set of points which form the perpendicular bisector of two given points.

Using the same idea, we define a line in taxicab geometry (or a taxicab bisector) as follows. If \( P_1 \) and \( P_2 \) are two distinct points in the plane and \( d_t \) is the taxicab distance function, then the set \( L = \{ P(x,y) \mid d_t(P, P_1) = d_t(P, P_2) \} \) is called the taxicab bisector of the line segment \( P_1P_2 \) or the taxicab bisector of the points \( P_1 \) and \( P_2 \).

Unlike lines in Euclidean coordinate geometry, which are characterized by slopes, taxicab bisectors (or lines in taxicab geometry) can be completely characterized into the following four forms. The description of their constructions will be given. Interested readers can refer to [4] for more details.

1. If \( P_1 \) and \( P_2 \) have either the same \( x \)-coordinates or the same \( y \)-coordinate, then the taxicab bisector of \( P_1 \) and \( P_2 \) is the same as the one in Euclidean geometry (see Figures 3).

2. If \( P_1 \) and \( P_2 \) have different \( x \) and \( y \) coordinates, then we construct the taxicab bisectors as follows.

Step 1. Draw vertical and horizontal lines through \( P_1 \) and \( P_2 \) to form a rectangle \( ABCD \) so that they are the opposite corners of a rectangle. For the sake of convenience, \( A \) is always the upper left-hand corner and \( C \) is the lower right-hand corner.

Step 2. (a) In the rectangle \( ABCD \), if \( |AB| > |AD| \), \( P_1 = B \), and \( P_2 = D \), then first locate the midpoints \( M \) and \( N \) of \( AB \) and \( AD \), respectively. Mark the point \( P \) which is \( |DN| \) units to the left of \( M \). Label \( Q \) as the point of intersection of the line through \( P \) with slope -1 and the line segment \( CD \). Let \( A_1 \) be the vertical ray above \( AB \) having \( P \) as its end point, let \( A_2 \) be the line segment \( P Q \), and let \( A_3 \) be the vertical ray below \( DC \) having \( Q \) as its end point. Then \( L \), which is the union of the sets \( A_1, A_2, \) and \( A_3 \), is the taxicab bisector of the points \( P_1 \) and \( P_2 \) (see Figure 4).

   (b) Similarly, if \( |AB| > |AD| \), \( P_1 = C \), and \( P_2 = A \), then we can construct \( B_1, B_2, \) and \( B_3 \) as in Part (a). Then \( L \), which is
the union of the sets $B_1, B_2,$ and $B_3,$ is the taxicab bisector of the points $P_1$ and $P_2$ (see Figure 5).

(c) If $|AB| < |AD|$, $P_1 = B$, and $P_2 = D$, then we can obtain the graph of the taxicab bisector of $P_1$ and $P_2$ by rotating the Figure 5 by 90° in the clockwise direction (see Figure 6A).

(d) In the rectangle $ABCD$, if $|AB| < |AD|$, $P_1 = C$, and $P_2 = A$, then we can obtain the graph of the taxicab bisector of $P_1$ and $P_2$ by rotating Figure 4 by 90° in the clockwise direction (see Figure 6B).

(e) If $|AB| = |AD|$, $P_1 = B$, and $P_2 = D$, then first obtain the region $F_1$ by shading the part which is on or to the left of $AP_2$ and on or above $AP_1$. Similarly, the region $F_2$ can be obtained by shading the region which is on or to the right of $P_1C$ and on or below $P_2C$. Finally, $F_3$ is the diagonal $AC$ of the square $ABCD$. Then the taxicab bisector of $P_1$ and $P_2$ is the union of the sets $F_1, F_2,$ and $F_3$ (see Figure 7).
(f) Similarly, if $|AB| = |AD|$, $P_1 = C$, and $P_2 = A$, then the taxicab bisector of $P_1$ and $P_2$ is the union of the sets $G_1, G_2$, and $G_3$ (see Figure 8).

![Figure 8]

Next, we would like to point out how one can find the shortest taxicab distance from a point $P(x_1, y_1)$ to a line of the form $Ax + By + C = 0$. If the absolute value of the slope of the line is greater than 1, then the horizontal distance from $P$ to the line will be the shortest taxicab distance. If the absolute value of the slope of the line is less than 1, then the vertical distance from $P$ to the line will be the shortest taxicab distance. If the slope is 1 or -1, then either the horizontal or the vertical distance from $P$ to the line will do.

**Remark.** Two taxicab bisectors $L_1$ and $L_2$ are said to be parallel if $L_1$ and $L_2$ do not intersect.

**Example 1.** Suppose $L_1$ is the taxicab bisector of the points $P_1(4, 3)$ and $P_2(0, 2)$, and $L_2$ is the taxicab bisector of the points $Q_1(10, 2)$ and $Q_2(6, 3)$.

Let $P(1.5, 4)$ and $Q(2.5, 1)$ be two points on $L_1$. Then $L_1$ and $L_2$ do not intersect, but the distance from $P$ to $L_2$ is 7 and the distance from $Q$ to $L_2$ is 5. Therefore, $L_1$ and $L_2$ are not equidistant (see Figure 9).

![Figure 9]

**Example 2.** Suppose $L$ is the taxicab bisector of the points $P_1(4, 0)$ and $P_2(0, -2)$, $L_1$ is the taxicab bisector of the points $R_1(7, 0)$ and $R_2(3, -2)$, and $L_2$ is the taxicab bisector of the points $Q_1(5, 0)$ and $Q_2(9, -2)$.

Consider $S(6, -2)$. Then $L$ and $L_1$ do not intersect, $L$ and $L_2$ do not intersect, $S$ is on both $L_1$ and $L_2$, and $L_1 \neq L_2$ (see Figure 10).

![Figure 10]

**Remark.** Example 2 indicates that the equivalent statements (2) through (4) in Euclidean geometry, which were mentioned in the introduction are invalid in taxicab geometry, and hence taxicab geometry is non-Euclidean.
REFERENCES

4. Y. Liu, Conic Sections in Taxicab Geometry, preprint, CMSU.